Derivatives of the Gaussian Free Field via Random Matrices

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Matrices

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Examples

Here are examples of 3×2 and 4×4 matrices:

$$\begin{pmatrix} 3 & -2 \\ e & 1 \\ -\pi & \sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{pmatrix}$$

Eigenvalues

This is how we multiply a vector by a matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 + a_{13}v_3 \\ a_{21}v_1 + a_{22}v_2 + a_{23}v_3 \\ a_{31}v_1 + a_{32}v_2 + a_{33}v_3 \end{pmatrix}$$

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Examples

$$\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 34 \end{pmatrix}$$

Eigenvalues

We say that $\lambda \in \mathbb{C}$ is an eigenvalue of a square matrix A if

$$Av = \lambda v$$

for some vector v. It turns out that there are n eigenvalues (up to multiplicity) of an $n \times n$ matrix A.

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Examples

$$\begin{pmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

so 3 is an eigenvalue of the original matrix.

Spectral Theorem

Examples

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If a matrix is symmetric and real, then all of its eigenvalues are real. Generally, we order the eigenvalues as follows:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n.$$

From now on, we only consider real symmetric matrices.

Random Variables

Define a probability density $p(x)$ to be a function

$$p : \mathbb{R} \to \mathbb{R}_{\geq 0}$$

such that $\int_{\mathbb{R}} p(x)dx = 1$.

Random Variables

A *random variable* X with values in $\mathbb{R}$ and density $p(x)$ is a “random number in $\mathbb{R}$ which can be sampled such that its frequency (histogram) as the number of samples increase converge to $p(x)$.”

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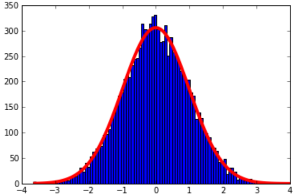
We say two random variables X and Y are independent if the outcome of X does not affect the outcome of Y and vice versa. For example, if X is the value of a flip of a coin, and Y is of another coin, then X and Y are independent. However, if X is the weather today, and Y is the weather tomorrow, then X and Y are not independent, i.e. correlated.

Example: Gaussian Random Variable

A *Gaussian Random Variable* is one that has

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Here is a sample of 10000 Gaussian random variables with $\mu = 0$ and $\sigma = 1$.



Random Vectors

Define a joint probability density $p(x)$ to be a function

$$p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$$

such that $\int_{\mathbb{R}^n} p(x) dx^n = 1$.

A random vector is a vector in $\mathbb{R}^n$ that takes random values with joint distribution $p(x)$.

Random Matrices

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We look at a special case, namely the Gaussian Orthogonal Ensemble, which is a Wigner matrix whose entries are Gaussian. Let X_N be an $N \times N$ GOE matrix.

Our Work

It was known that the eigenvalues of X_N converge to the Gaussian Free Field as $N \to \infty$. Letting $\lambda_1, \dots, \lambda_N$ be the eigenvalues of X_N, it suffices to study

$$\lambda_1^k + \cdots + \lambda_N^k = \mathrm{tr} X_N^k$$

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We looked at a "discrete derivative" of X_N, which means we looked at the eigenvalues of X_N, along with the eigenvalues $\mu_1, \ldots, \mu_{N-1}$ of the submatrix X_{N-1}. Again, it suffices to study

$$\lambda_1^k + \cdots + \lambda_N^k - \mu_1^k - \cdots - \mu_{N-1}^k = \mathrm{tr} X_N^k - \mathrm{tr} X_{N-1}^k$$

for all positive integers k. We found that this did converge to the derivative of the GFF.

Combinatorics

We can expand the trace in terms of the entries as

$$\mathrm{tr} X_N^k = \sum_{i_1,\ldots,i_k=1}^{N} X_N(i_1, i_2) X_N(i_2, i_3) \cdots X_N(i_k, i_1).$$

Then,

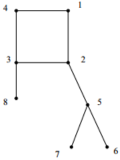
$$\mathbb{E}\, \mathrm{tr} X_N^k = \sum_{i_1,\ldots,i_k=1}^{N} \mathbb{E}\, X_N(i_1, i_2) X_N(i_2, i_3) \cdots X_N(i_k, i_1).$$

Now, all results reduce to combinatorics of graphs constructed from $(i_1, \ldots, i_N)$.

- Vertices are $\{i_1, \ldots, i_k\}$
- Edges are $\{\{i_1, i_2\}, \{i_2, i_3\}, \ldots, \{i_k, i_1\}\}$

Unicyclic Graphs

Only *unicyclic* graphs contribute in the limit $N \to \infty$:



These graphs have the same number of vertices and edges.

Discrete Derivative Combinatorics

- Note that

$$
\begin{aligned}
\mathrm{tr} X_N^k - \mathrm{tr} X_{N-1}^k &= \sum_{i_1,\ldots,i_k=1}^{N} X_N(i_1,i_2)X_N(i_2,i_3)\cdots X_N(i_k,i_1) \\
&- \sum_{i_1,\ldots,i_k=2}^{N} X_N(i_1,i_2)X_N(i_2,i_3)\cdots X_N(i_k,i_1) \\
&= \sum_{\substack{i_1,\ldots,i_k=1 \\ \exists j \text{ s.t } i_j=1}}^{N} X_N(i_1,i_2)X_N(i_2,i_3)\cdots X_N(i_k,i_1).
\end{aligned}
$$

- Our corresponding graph is rooted at 1: it must contain the vertex 1.

Future Directions

The main idea is to look at higher discrete derivatives. However, we have reason to believe that the mth discrete derivative of X_N converges to the mth derivative of the GFF for $m \geq 2$, but these derivatives are infinite. This has to do with the fact that after taking a derivative of the GFF, the elements of the GFF become "too independent" of one another.

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Another possible direction is to look at edge results of the eigenvalues after taking a discrete derivative. This has to do with looking at the largest eigenvalues of the random matrices, and understanding their statistical properties.

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